CHAPTER VII

Theoretical Analysis of Modes of Vibration for Isotropic Rectangular Plates Having All Surfaces Free

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7.1. Introduction

The comparatively recent advent of crystal controlled oscillators and of wave filters employing piezoelectric elements has resulted in an extensive study of the ways in which plates made of elastic materials such as quartz or rochelle salt can vibrate. Of special interest bave been the resonant frequencies associated with these modes of motion. As will be indicated in subsequent paragraphs, the general solution to tbe problem of greatest interest is quite complex, and has not been forthcoming, (i.e., as applied to rectangular plates completely unrestrained at all boundary surfaces). For this reason numerous approximate solutions have been developed which yield useful information in spite of their limitations. Several of these solutions will be discussed in the following sections. The three general types of modes (i.e., the extensional, sbear, and flexural) will be analyzed in some detail. Also, as a preliminary step the formulation of the general problem along classical lines will be developed.

For the most part, the solutions obtained here are limited to those for an isotropic body. However, such solutions provide considerable guidance for the modes of motion existing in an aeolotropic body such as quartz.

7.2. Method of Analysis

In order to set up the desired matbematical statement of our problem it will be necessary to consider first of all two very fundamental relationsbips. The first of these is the well known law of Newton which states that a force f acting on a mass m produces an acceleration a in accordance with the formula

$$f = m \cdot a$$

The second relationship which we shall need is Hooke's law relating the strains in a body to the stresses. If forces are applied to the ends of a long slender rod made of an elastic material such as steel (Fig. 7.1) a certain amount of stretching takes place. If tbe forces are not too great, a linear

relationship between the applied stress and ensuing strain is found to exist. Expressed as an equation

$$\frac{X_x}{x_x} = E \quad \text{in which } X_x \text{ is the force per unit area,}$$

x_x is the strain per unit length, and E is a constant known as Young's Modulus. (Refer to Section 7.7 for further definition of terms).

In an analogous manner, shearing stresses applied to an elastic solid as shown by Fig. 7.2 produce a shearing strain such that

$$\frac{X_y}{x_y} = A, \text{ the shear modulus.}$$

In general there will be contributions to a particular strain from any of the stresses which may happen to exist. For example, when an isotropic

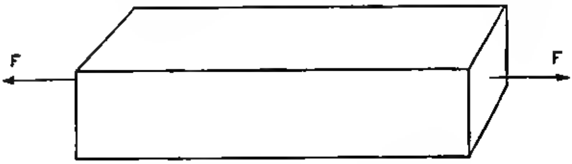

Fig. 7.1—Bar under tensional stress

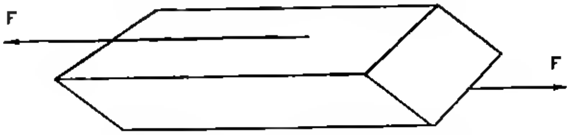

Fig. 7.2—Bar under shearing stress

bar is stretched, there will be a contraction along the width which has been produced by a stress along the length. A statement of these relationships (known as Hooke's Law) is given by the equations of Section 7.8.

It is now of interest to consider the conditions of equilibrium for a very small cube cut out of the elastic medium which in general is stressed and in motion. Reference to Fig. 7.3 will help to visualize the stresses which may exist on the faces of this cube. Since these stresses vary continuously within the medium, a summation of the forces acting on the cube along each of the major axes can be made with the use of differential calculus. From Newton's Law previously cited, it is apparent that any unbalance of these forces will result in an acceleration inversely proportional to the mass of our small cube. Three equations may then be derived, one for each major direction.[1] If only simple harmonic motion is considered (i.e. all displace-

[1] Refer to "Theory of Elasticity" by S. Timoshenko or to any standard text on elasticity.

ments are proportional to sin ωt where $\omega = 2\pi$ times frequency) the following simplified equations result.

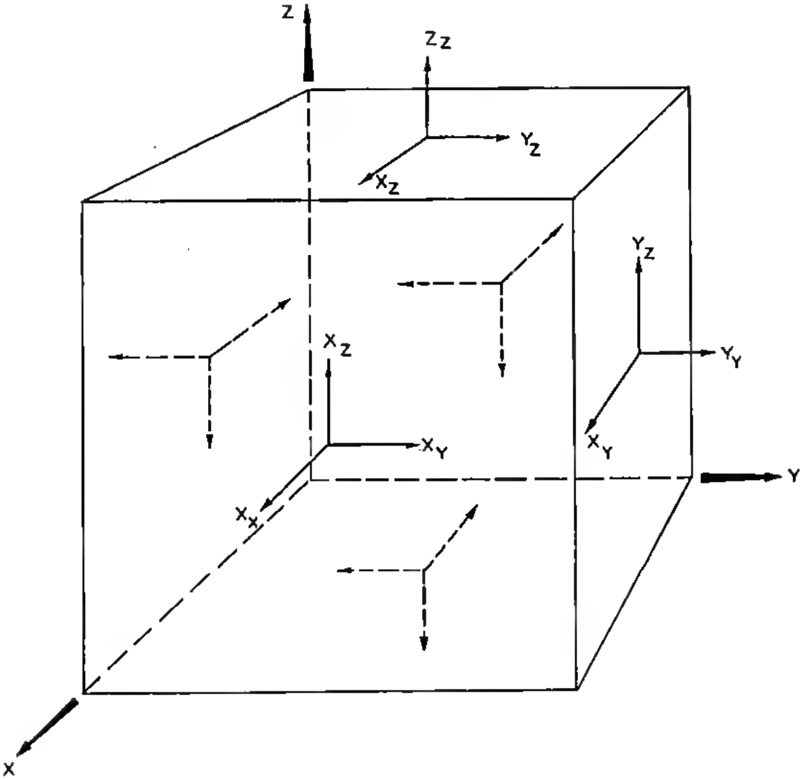

Fig. 7.3—Stresses acting on small cube

$$\left.\begin{aligned}\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} &= -\rho\omega^2 u \\ \frac{\partial Y_y}{\partial y} + \frac{\partial X_y}{\partial x} + \frac{\partial Y_z}{\partial z} &= -\rho\omega^2 v \\ \frac{\partial Z_z}{\partial z} + \frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} &= -\rho\omega^2 w\end{aligned}\right\} \qquad (7.1)$$

Since stresses are related to strains in a very definite manner, the above equations may be converted into a more useful form involving only displacements. For *isotropic media*, the following results.

$$\left.\begin{aligned} A\nabla^2 u + B\frac{\partial \epsilon}{\partial x} &= -\rho\omega^2 u \\ A\nabla^2 v + B\frac{\partial \epsilon}{\partial y} &= -\rho\omega^2 v \\ A\nabla^2 w + B\frac{\partial \epsilon}{\partial z} &= -\rho\omega^2 w \end{aligned}\right\} \tag{7.2}$$

In this grouping,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\epsilon = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

and A and B are given in terms of the fundamental elastic constants λ and μ with $A = \mu$, $B = \lambda + \mu$.

An even more elegant statement of the equilibrium conditions attributable to Love[2] follows immediately from equations 7.2, since by differentiating each one in turn with respect to x, y, and z respectively, and then adding results, one obtains the wave equation

$$(\nabla^2 + h^2)\epsilon = 0 \tag{7.3}$$

where

$$h^2 = \frac{\rho\omega^2}{A + B} = \frac{\rho\omega^2}{\lambda + 2\mu}$$

Whatever our solution may be, then, it must satisfy equation (7.3). If such an expression for ϵ is found, the displacements formed in the following way will satisfy equations 7.2 as can be shown by direct substitution.

$$u = -\frac{1}{h^2}\frac{\partial \epsilon}{\partial x} \qquad v = -\frac{1}{h^2}\frac{\partial \epsilon}{\partial y} \qquad w = -\frac{1}{h^2}\frac{\partial \epsilon}{\partial z} \tag{7.4}$$

In addition to equations 7.2, another set of requirements will be necessary when any particular problem is considered. They are known as the boundary conditions, and in general are easily deduced from a knowledge of how the plate or bar is held.

For a rectangular plate free on all surfaces, the boundary condition is simply that all surface tractions vanish. This requires certain stresses to become zero at the boundary as can be seen from the following expressions for the x, y, and z components of traction in terms of unit stresses.

[2] A E. Love, "A Treatise on the Mathematical Theory of Elasticity."

$$\left.\begin{aligned}\bar{X} &= X_x\ell + X_y m + X_z n \\ \bar{Y} &= Y_y m + Y_z n + X_y \ell \\ \bar{Z} &= Z_z n + X_z \ell + Y_z m\end{aligned}\right\} = 0 \text{ for free surfaces} \tag{7.5}$$

(ℓ, m, and n are direction cosines of the normal to the surface at tbe point in question).

The general problem is now seen to be one of finding solutions for tbe displacements u, v, and w sucb that both tbe equilibrium and boundary conditions are satisfied. In the following section several interesting solutions will be considered for rectangular plates baving all surfaces free, tbis being the case of greatest interest in so far as tbis paper will be concerned.

7.3. Extensional Vibrations

One of the most useful modes of vibration of practical interest is tbe extensional, in wbich particle motion takes place in essentially one direction so as to alternately stretcb and compress the elastic medium. Piezoelectric

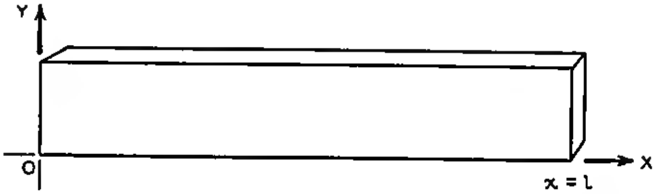

Fig. 7.4—Longitudinal bar

plates vibrating in this manner, and of the shapes shown in figures 7.4 and 7.5 have been used extensively in wave filter and oscillator circuits. The approximate resonant frequencies corresponding to this type of motion are easily obtained by a consideration of equations 7.1 and 7.2. For tbe longitudinal bar of Fig. 7.4 the only stress that need be considered is the X_x extensional, all otber stresses being so small tbat tbey can be neglected. The equilibrium equation then becomes

$$\frac{\partial X_x}{\partial x} = -\rho\omega^2 u \tag{7.6}$$

or, since

$$\frac{\partial u}{\partial x} = \frac{1}{E} X_x$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\rho}{E}\omega^2 u \tag{7.7}$$

It is easily seen that $u = \cos kx$ is a solution to this equation if $k = \omega\sqrt{\frac{\rho}{E}}$. If now the boundary condition that the stress X_x must become zero at the ends of the bar (i.e., $x = 0$, $x = \ell$ — refer to Eq. (7.5)), is fulfilled, the solution will be complete. At $x = 0$, $X_x = E\frac{\partial u}{\partial x}$ will always equal zero. Furthermore, if $k = \frac{\pi}{l}$ or any whole number multiple of $\frac{\pi}{\ell}$ the extensional stress will likewise reduce to zero at $x = \ell$. The desired solution will then be as follows, f being the resonant frequencies.

$$\left.\begin{aligned} u &= \cos \omega \sqrt{\frac{\rho}{E}}\, x \\ \omega &= 2\pi f = \frac{m\pi}{\ell}\sqrt{\frac{E}{\rho}} \\ m &= 1, 2, 3, \text{etc.} \end{aligned}\right\} \tag{7.8}$$

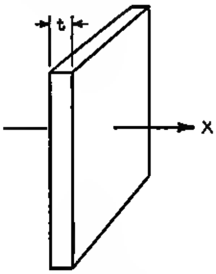

Fig. 7.5—Thin plate

The plate of Fig. 7.5 will now be considered. Here it can no longer be assumed that the X_x stress is the only one of importance. Instead, the displacements v and w will be considered zero and the displacement u a function of x only. This means that the shear stresses X_y, X_z, Y_z vanish, so that the equilibrium equations 7.2 reduce to

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x^2} = -\rho\omega^2 u \tag{7.9}$$

or

$$\frac{\partial^2 u}{\partial x^2} = \frac{-\rho\omega^2 u}{A + B} \tag{7.10}$$

This is seen to be of the same form as equation (7.7) previously discussed, and will again have the solution $u = \cos kx$ with $k = \omega\sqrt{\frac{\rho}{A + B}}$. The

boundary condition on the X_x stress will be met if $k = \frac{m\pi}{t}$ so that the following solutions result.[3]

$$
\begin{aligned}
u &= \cos \omega \sqrt{\frac{\rho(1+\sigma)(1-2\sigma)}{E(1-\sigma)}}\, x = \cos \omega \sqrt{\frac{\rho}{\lambda + 2\mu}}\, x \\
\omega &= 2\pi f = \frac{m\pi}{t} \sqrt{\frac{E}{\rho} \cdot \frac{(1-\sigma)}{(1+\sigma)(1-2\sigma)}} = \frac{m\pi}{t} \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad (7.11) \\
m &= 1, 2, 3, \text{etc.}
\end{aligned}
$$

It is seen that this formula for resonant frequencies is the same as given by equations 7.8, with E replaced by $\frac{E \cdot (1-\sigma)}{(1 - 2\sigma^2 - \sigma)}$, so tbat the frequency constant $f \cdot t$ will be somewhat higher than $f \cdot \ell$ for a long slender bar.

It is recognized that the solutions derived above hold true only for the limiting cases of a long slender bar, and a very thin plate respectively. It is therefore of interest to trace the resonant frequencies corresponding to these extensional modes of vibration as departure is made from the limiting cases mentioned above.

An experimental plot of the resonant frequencies of a thin plate of length ℓ and width w reveals that the frequency of the longitudinal mode first discussed is gradually lowered as the width of the plate is increased. There is also another frequency corresponding to an extensional vibration along the width which for a very narrow plate corresponds to the second type of extensional mode considered in the foregoing paragraphs, except that the frequency constant will be slightly different because coplanar stresses are involved.

As seen from Fig. 7.6, the resonant frequency curves do not cross, but exhibit coupling effects. This is understandable from the fact that motion in one direction is mechanically coupled to motion in the other as indicated by Poisson's ratio σ.

In order to derive expressions for the u and v displacements associated with the extensional mode along the length, taking into account the above coupling effect, the following analysis proves interesting.

Consider the infinite isotropic strip of width b as shown by figure 7.7. As will be demonstrated presently, solutions can be found such that the equilibrium equations and the boundary conditions are precisely satisfied. Furthermore it will be found possible to cut a section out of this strip in

[3] If the length and width of the plate are very large in comparison to the thickness, the boundary conditions for the Y_y and Z_z stresses may be neglected without causing appreciable error. The quantity $A + B$ has been evaluated in terms of E and σ for purposes of comparison.

such a way that the boundary conditions for the cut edges are very nearly satisfied. The plate formed in this way may then be considered as vihrating at the required frequency f, which will then he the resonant frequency desired.

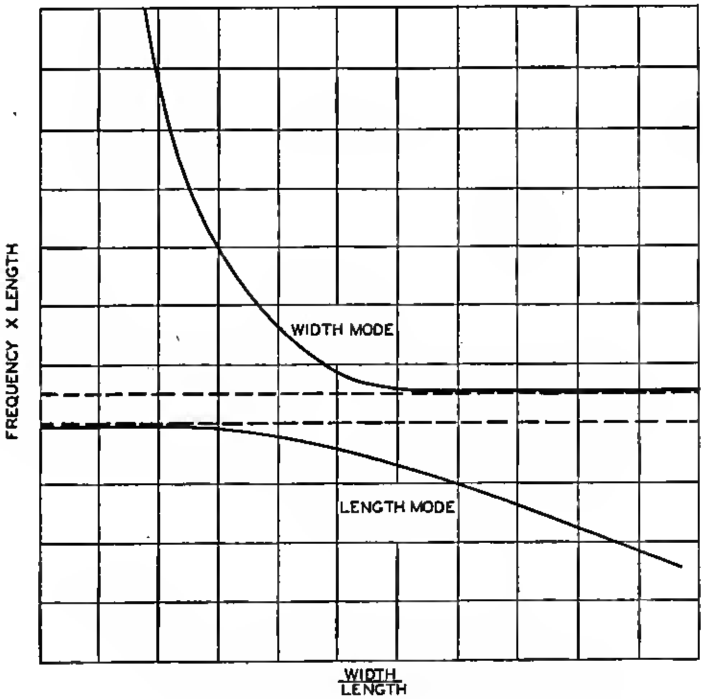

Fig. 7.6—Extensional modes with mechanical coupling

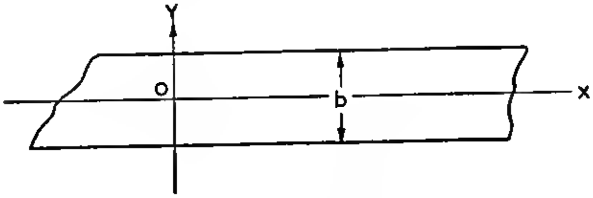

Fig. 7.7—Infinite strip

Let displacements he arhitrarily chosen in the following way:

$$\left.\begin{aligned} u &= U \cos kx \cos \zeta y \\ v &= V \sin kx \sin \zeta y \end{aligned}\right\} \qquad (7.12)$$

As shown in Section 7.9, two solutions of this type will satisfy tbe equilibrium equations precisely. One corresponds to $\epsilon = 0$ in the wave equation 7.3, wbile for the other $\epsilon \neq 0$. Superposition of tbese two solutions and proper evaluation of parameters make it possible to satisfy the boundary conditions at the edge of the strip; namely, that at $y = \pm \frac{b}{2}$, $Y_y = 0$ and $X_y = 0$. (Refer to equations 7.5). The following transcendental equation is obtained

$$\frac{\cot \ell_1 \frac{b}{2}}{\cot \ell_2 \frac{b}{2}} = \frac{-2\ell_1 \ell_2 k^2(1 - \sigma)}{(\ell_2^2 - k^2)(\ell_1^2 + \sigma k^2)} \tag{7.13}$$

in which

$$\left.\begin{aligned} \ell_1^2 &= \frac{A}{A + B}\theta^2 - k^2 \\ \ell_2^2 &= \theta^2 - k^2 \\ \theta^2 &= \frac{\rho\omega^2}{A} \end{aligned}\right\} \tag{7.14}$$

This equation may be solved graphically to yield values of frequency corresponding to given values of k. For our discussion of the length extensional mode of vibration, the first root only will be considered.

Fig. 7.8 shows a plot of $\theta \cdot b$ against $b \cdot k$ assuming that Poisson's ratio is .33.[4] If $k = 1$, and $b = 1$, for example, $\theta = \sqrt{\frac{\rho}{A}}\,\omega = 1.62$.

The equations for the displacements when determined as explained in Section 7.9 become:

$$\begin{aligned} u &= U_1 [\cosh \cdot 344\, y + .402 \cos 1.278\, y] \cos x \\ v &= U_1 [.344 \sinh .344\, y + .315 \sin 1.278\, y] \sin x \end{aligned} \tag{7.15}$$

All three stresses X_x, Y_y, and X_y may be calculated from tbe above equations. If the length of our plate is made equal to $m\pi$, wbere m is an integer, the extensional stress X_x will equal zero regardless of y at the boundaries $x = 0$ and $x = \ell$ since $X_x \alpha \sin x = 0$ when $x = m\pi$. Also it can be shown by calculation that X_y is so small in comparison to tbe extensional stresses as to be entirely negligible; hence our solution is complete.

If $\ell = \pi$, the plate will be vibrating in its fundamental longitudinal mode. The distortion whicb results is sbown by Fig. 7.9. It is seen tbat most of

[4] Plotted in this way, the same curve results regardless of the value of b chosen for the purpose of solving Eq. 7.13.

the motion is along the x axis, though there is a certain amount of lateral contraction as the plate elongates.

The second harmonic will have the same resonant frequency if $\ell = 2\pi$, the third if $\ell = 3\pi$, etc.

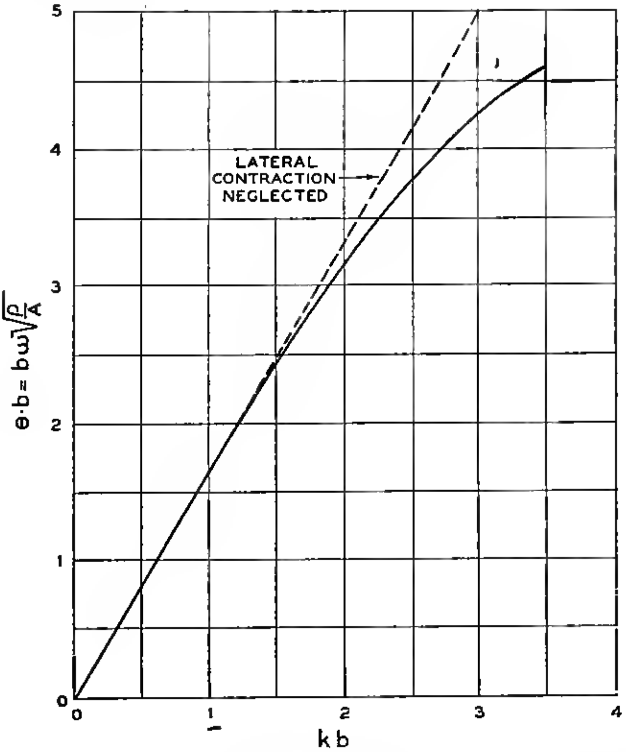

Fig. 7.8—$\theta \cdot b$ versus $k \cdot b$ for plate longitudinal modes $\left(\sigma = .33,\ k = \frac{m\pi}{\text{length}}\right)$

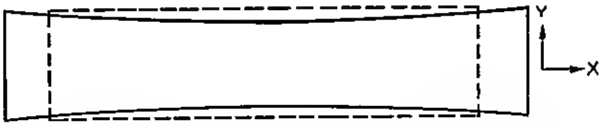

Fig. 7.9—Distortion of plate vibrating in first longitudinal mode

In addition to harmonic modes along the length just considered there will be those for which the motion breaks up along the width. In general, the distortion of the plate may be quite complex with simultaneous variations along both dimensions. Similarly, for plates such as shown in Fig. 7.5

there will be many extensional modes which have resonant frequencies somewhat above those given by Eq. 7.11. Analysis of the motion shows that for these modes the displacement along the thickness varies periodically (or "breaks up") along the major dimensions of the plate. There again the distortion pattern of the plate may become very complex.

7.4. Shear Vibrations

The second class of vibrations which will now be considered is the shear. This type of mode is of special importance because of the fact that piezoelectric plates vibrating in shear are widely used for frequency control of oscillators. For example, the AT quartz plate which is so much in demand utilizes a fundamental thickness shear mode in which particle motion is principally at right angles to the thickness. The distortion of the plate will be similar to that shown in Fig. 7.2.

A simple, yet very useful formula for the resonant frequencies associated with the above type of displacement has been derived on the assumption

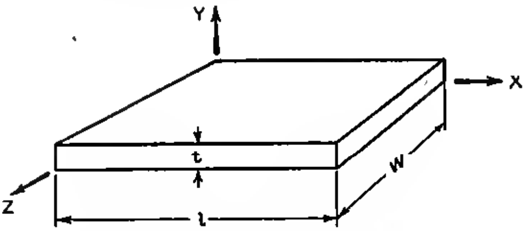

Fig. 7.10—Orientation of thin plate

that the length and width of the plate are very large in comparison to the thickness. For the xy shear mode, the displacement u is assumed to be $u = U \cos ky$, all other displacements being equal to zero. The only stress that need be considered then, is the X_y shear which is proportional to sin ky. Boundary conditions on this stress at the major surfaces of the plate are easily satisfied by choosing k such that $X_y = 0$ at $y = 0$ and $y = t$. (Refer to Fig. 7.10.) This will be the case if $k = \dfrac{m\pi}{t}$, where m is any integer, and t is the thickness of the plate. By using the simplified equilibrium equation as reduced from equations 7.1, a formula for the resonant frequencies is obtained in much the same manner as for extensional thickness modes.

$$\omega = 2\pi f = \frac{m\pi}{t}\sqrt{\frac{A}{\rho}} \qquad m = 1, 2, 3, \text{etc.} \tag{7.16}$$

In this formula the shear modulus A appears instead of Young's modulus as in the case of longitudinal modes. Harmonic modes are given by values of m greater than unity.

In addition to the resonant frequencies predicted hy the foregoing analysis, there will he others corresponding to shear vihrations in which the principal shear stress varies periodically along the length and width of the plate. A formula which yields the approximate frequencies for these modes is developed in Section 7.9. It is shown that if the length and width are large in comparison to the thickness, the following expression may he used:

$$\omega = 2\pi f = \pi \sqrt{\frac{1}{\rho}} \sqrt{c_{11}\frac{n^2}{\ell^2} + \frac{c_{66}m^2}{\ell^2} + \frac{c_{55}p^2}{w^2}} \tag{7.17}$$

In this formula which has heen derived for xy shears the c constants are the standard elastic constants for aeolotropic media. For isotropic plates such as have heen considered up to this point

$$c_{11} = \frac{E(1-\sigma)}{1 - 2\sigma^2 - \sigma} = \lambda + 2\mu$$

and

$$c_{55} = c_{66} = A, \text{ the shear modulus} \tag{7.18}$$

Various comhinations of the integers m, n, and p may he chosen, with the restriction that neither m nor n can equal zero. It is seen that if ℓ and w are very large the formula reduces to that of Eq. 7.16 which was derived on precisely that hasis. Also, it is seen that the more complex modes all lie somewhat ahove the fundamental shear ohtained hy setting $m = n = 1$ and $p = 0$.

Plate shear modes are also of considerahle interest, particularly the one of lowest order. For a plate having a large ratio of length to width a formula similar to that given hy equation 7.17 (hut for two dimensions only) may he developed. If the plate is nearly square, however, this formula no longer yields sufficiently accurate values for the resonant frequencies. Coupling to other modes of motion[5] complicates the prohlem so much that only experimental results have heen of much practical consequence. Fig. 7.11 shows in an exaggerated way the distortion of a nearly square plate vihrating in the first shear mode.

7.5. Flexural Vibrations

7.51. *Plate Flexures*

One of the most studied types of vihrations has heen the flexural. Perhaps this is true hecause it is the most apparent and comes within the realm of experience of nearly everyone. The phenomena of vihrating reeds, xylophone hars, door hell chimes, tuning forks, etc. are quite well known.

[5] It is found experimentally that odd order shears are strongly coupled to even order flexures; similarly, even order shears and odd order flexures are coupled.

Beam theory has been used quite extensively to derive tbe equations which yield the resonant frequencies and displacements for bars vibrating in flexure. To obtain reasonably accurate results for ratios of width to length approaching unity, however, the effects of lateral contraction, rotary inertia, and shearing forces must be considered. This leads to a rather complicated solution whicb is much more accurate than that derived by the use of simple beam theory only, though it is still approximate in nature.

For two dimensional plates free on all edges a method of analysis may be used which is similar to that described under extensional modes. While it is somewbat involved it yields direct expressions for the two displacements u and v, so that all stresses may be calculated, and the extent to which boundary conditions are satisfied determined.[6]

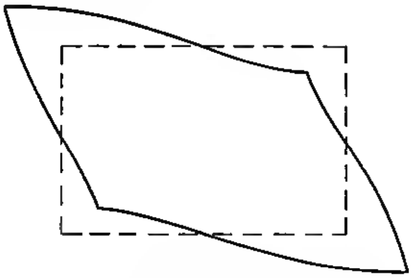

Fig. 7.11—Distortion of plate in first shear mode

Solutions for u and v are assumed to be of the form

$$\begin{aligned} u &= U \sin \ell y \cos kx \\ v &= V \cos \ell y \sin kx \end{aligned} \tag{7.19}$$

For the infinite strip previously considered a transcendental equation is obtained which is the same as equation 7.13 with the exception that the left-hand expression is inverted.

$$\frac{\tan \ell_1 \frac{b}{2}}{\tan \ell_2 \frac{b}{2}} = \frac{-2\ell_1 \ell_2 k^2(1 - \sigma)}{(\ell_2^2 - k^2)(\ell_1^2 + \sigma k^2)} \tag{7.20}$$

(Refer to Eq. 7.14 also.)

[6] This is an extension of Doerffler's analysis used to obtain harmonic flexure frequenceis for plates—"Bent and Transverse Oscillations of Piezo-Electrically Excited Quartz Plates"—Zeitschrift Für Physik, v. 63, July 7, 1930, p. 30. Also refer to "The Distribution of Stress and Strain for Rectangular Isotropic Plates Vibrating in Normal Modes of Flexures"—New York Univ. Thesis by Author, June 1940.

The lowest order solution to this equation is found to correspond to flexure vibrations in the infinite strip. A calculation of stresses, however, reveals that boundary conditions cannot be satisfied properly even for the case of a long narrow plate. It can be shown, however, that another solution may be derived for the same value of frequency by letting k become imaginary. This simply means that the u and v displacements become hyperbolic functions of x instead of sinusoidal. The two complete solutions for the infinite strip may then be superimposed and parameters adjusted so that for definite values of length corresponding to fundamental and harmonic modes the proper stresses reduce essentially to zero on the ends of the plate. For plates having a ratio of width to length less than .5, this method gives very accurate expressions for displacements and stresses. If only the resonant frequency is required, ratios up to unity and beyond (for the fundamental mode) may be considered.

An example has been worked out to provide a complete picture of the displacements for a bar of width = 1, $k = 1$ and $\sigma = .33$. Use of equation 7.20 yields the quantity $\theta^2 = \dfrac{\rho\omega^2}{A} = .166$ from which the resonant frequency may be obtained. Using this value of θ^2, one finds that $k^2 = -.800$ also satisfies equation 7.20. By making the total length of the bar equal to 4.50 the X_x extensional stress and the X_y shear stress may be made essentially zero on the ends of the plate regardless of y.[7]

The following expressions for u and v are obtained:

$$\begin{aligned} u &= (\sinh .9132\, y - 1.02 \sinh .9718y) \sin x \\ &\quad -.160\, (\sin .9828y - .9568 \sin .9250y) \sinh .8944x \\ v &= (-1.094 \cosh .9132y + .9915 \cosh .9718) \cos x \\ &\quad -.160\, (.9095 \cos .9828y - .990 \cos .9250y) \cosh .8944x \end{aligned} \tag{7.21}$$

Fig. 7.12 shows the distortion of the plate as calculated from the above expressions. It is seen that there will be two points at which there is no motion in either the x or y directions. These nodal points can be used in holding the plate, since it may be clamped firmly there without altering the displacements or resonant frequency. For the example shown, these nodes are positioned a distance of $.211\ell$ from the ends of the plate as compared to $.224\ell$ for a long thin bar.

[7] A graphical solution to determine ℓ is most convenient in which parameters are adjusted so that $X_x = 0$ at $x = \pm\frac{\ell}{2}$ and $y = \pm\frac{b}{2}$; $X_y = 0$ at $x = \pm\frac{\ell}{2}$ and $y = 0$. These stresses will remain essentially zero for all values of y if the ratio of $\frac{w}{\ell}$ is not too great.

Figures 7.13 and 7.14 show the distribution of the principle stresses as a function of position along the length. It is seen that for the particular

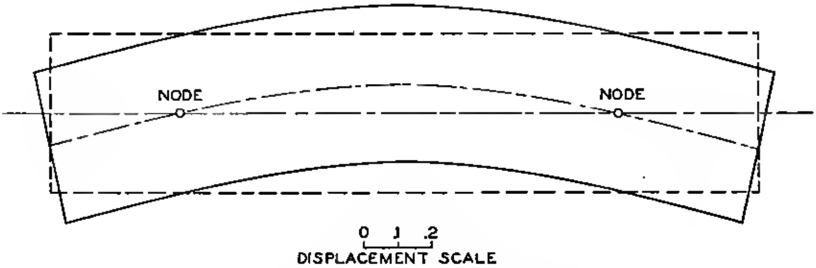

Fig. 7.12—Distortion of bar vibrating in first free-free flexure mode

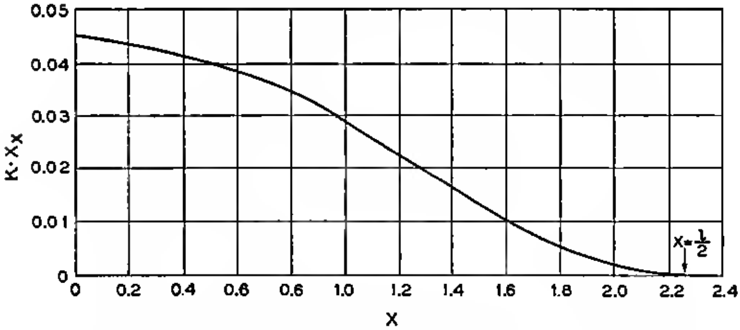

Fig. 7.13—Distribution of longitudinal stress for free-free bar vibrating in first flexure mode

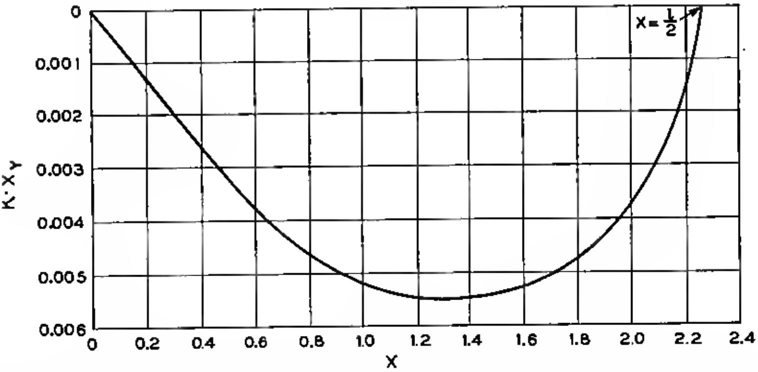

Fig. 7.14—Distribution of shear stress for free-free bar vibrating in first flexure mode

example cited, the maximum shear stress is only about one-tenth the maximum X_x extensional stress. Both of these stresses reduce to zero at

the ends of the plate as they should in order to satisfy the boundary conditions. As tbe ratio of $\frac{w}{\ell}$ is increased tbe sbear stress becomes of greater importance.

7.52. *Thickness Flexures*

The final analysis to he considered in this paper is for thickness flexures along the width or length of a thin plate. These modes are of particular interest in connection with the dimensioning of quartz plates for which it is desirable to utilize the fundamental thickness shear mode. (AT plate, for example.) It is found experimentally that even ordered thickness flexures are coupled to this shear to such a degree that at certain ratios of dimensions the operation of the plate as an oscillator or filter component is impaired.

The two-dimensional solution derived in the preceding paragraphs can he used to predict certain harmonic thickness flexures; however, in order to obtain a complete picture it is necessary to extend the theory to three dimensions. This has been done by the author with the following transcendental equation as a result (refer to Section 7.93).

$$\frac{\tan \ell_1 \frac{b}{2}}{\tan \ell_2 \frac{b}{2}} = \frac{-2\ell_1 \ell_2 A\alpha^2}{[\sigma B(\ell_1^2 + \alpha^2) + A\ell_1^2][\ell_2^2 - \alpha^2]} \tag{7.22}$$

Solutions to this equation are exact in nature for a plate of thickness b and of infinite extent in both the x and z directions. The quantity α^2 is equal to the sum of the squares of k and m which appear in the expressions for displacements as follows:

$$\left.\begin{aligned} u &= U f_1(y) \sin kx \cos mz \\ v &= V f_2(y) \cos kx \cos mz \\ w &= W f_3(y) \cos kx \sin mz \end{aligned}\right\} \tag{7.23}$$

Also in equation (7.22)

$$\left.\begin{aligned} \ell_1^2 &= \theta^2 \frac{A}{A+B} - \alpha^2 \\ \ell_2^2 &= \theta^2 - \alpha^2 \\ \theta^2 &= \frac{\rho\omega^2}{A} \end{aligned}\right\} \tag{7.24}$$

The lowest order solution to equation (7.22) with α^2 positive again corresponds to flexure vibrations, as in the two dimensional case. Fig. 7.15 shows a plot of $\theta \cdot b$ against $\alpha \cdot b$ calculated for $\sigma = .3$.

For reasonably higb order flexures it may be reasoned tbat the true displacements will be very nearly the same as tbose for tbe doubly infinite plate as derived by the above method since the correction necessary to fulfill the boundary conditions will only apply very close to the edges of the plate. It will then be sufficient to choose values for k and m such that $k = p\frac{\pi}{l}$ and $m = \frac{q\pi}{w}$ where p and q are integers. The values of α^2 ohtained in this way determine the corresponding resonant frequencies.

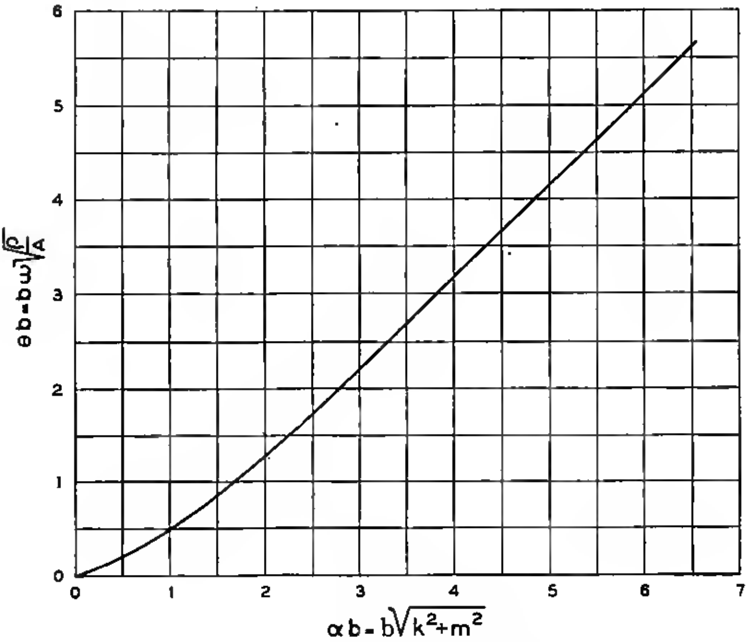

Fig. 7.15—$\theta \cdot b$ versus $\alpha \cdot b$ for thickness flexures

If it is desired to solve for the ordinal xy flexures, for example, m should he set equal to zero. The displacements in this case will be independent of the z dimension. When q is assigned values other than zero however, the resulting modes may be considered as xy flexures which vary or break up along the third major dimension. If q is small the resonant frequencies will lie only slightly higher than that of the corresponding ordinal flexure for which $q = 0$.

Fig. 7.16 shows a few of the resonant frequencies as calculated for values of shear modulus and density corresponding to AT quartz. Tbe effects of coupling to the fundamental thickness shear are shown by dotted lines for the 14th xy flexure. As might be expected there is similar coupling between

the 14th flexure which breaks up once along the z dimension and the shear which breaks up once along z—etc. A few of these flexures which break up along z are shown for the 16th ordinal flexure.

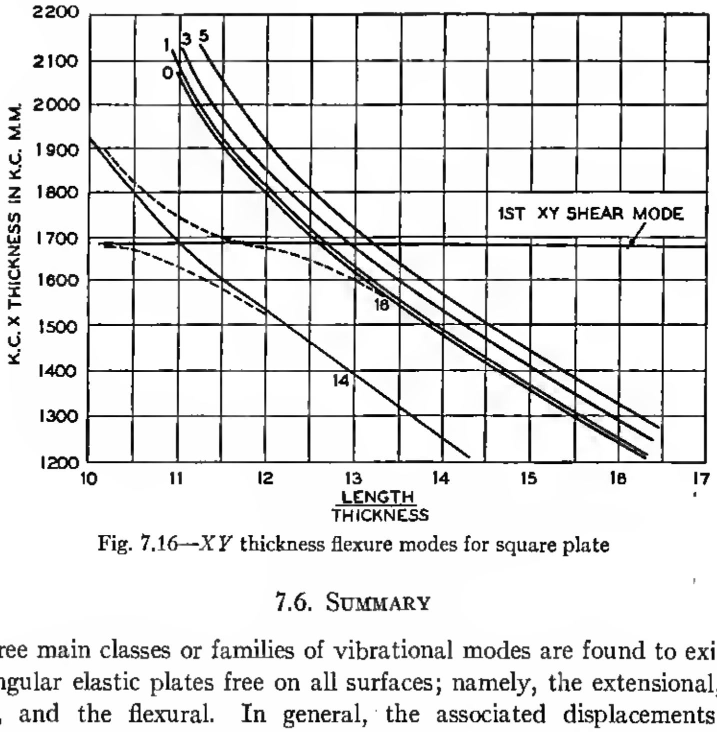

Fig. 7.16—XY thickness flexure modes for square plate

7.6. Summary

Three main classes or families of vibrational modes are found to exist in rectangular elastic plates free on all surfaces; namely, the extensional, the shear, and the flexural. In general, the associated displacements are functions of all three dimensions and may vary in such a manner as to make the distortion of such plates quite complex.

For certain limiting cases, approximate solutions for the resonant frequencies and displacements (from which strains and stresses may be calculated) can be derived. Though there are a number of methods that can be used for specific problems, it has been found very convenient to utilize the classical formulation. For this reason the basis of this method has been discussed briefly. In essence it requires that displacements and stresses occurring within the elastic solid satisfy conditions of equilibrium as derived from Newton's Law. At the boundaries, certain other relations must be satisfied in order that conditions of clamping might be fulfilled. For plates entirely unrestrained the latter requires that all forces (tractions) acting through the free surfaces must vanish.

For thin rectangular plates (such as quartz crystal oscillator plates) the modes of greatest practical consequence are plate modes, for which all

stresses are essentially coplanar and independent of the thickness, and thickness modes, for which all dimensions must be considered except in limiting cases.

Because of their great utility, simplified formulae have been derived for the resonant frequencies associated with long, narrow bars vibrating longitudinally, thin plates with extensional motion along the thickness dimension, and thin plates vibrating with shearing motion at right angles to the thickness.

Exact solutions for the infinite strip have been derived, and used in obtaining the displacements and resonant frequencies for flexural and longitudinal modes. Such solutions take account of the fact that the width of the plate may become appreciable. While limiting cases of plate shear may be analyzed, solutions for ratios of $\frac{w}{\ell}$ approaching unity have not proved very satisfactory. This is attributable to the fact that coupling to flexural modes is severe.

Thickness flexural modes which exhibit displacement variations along both length and width dimensions of the plate have been analyzed by extending the "infinite strip" theory to three dimensions. Solutions obtained are fairly accurate if the harmonic order of the flexure is sufficiently great.

7.7. Nomenclature

ρ = density

E = Young's modulus

σ = Poisson's ratio

$$A = \text{Shear modulus} = \frac{E}{2(1 + \sigma)} = \mu$$

$$B = \frac{E}{2(1 + \sigma)(1 - 2\sigma)} = \lambda + \mu \quad \text{for 3 dimensions}$$

$$= \frac{E}{2(1 - \sigma)} \text{ for plane stress}$$

ω = angular velocity = $2\pi f$

$$\theta^2 = \frac{\rho\omega^2}{A}$$

u, v, w = displacements in x, y and z directions

$$\epsilon = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\nabla^2 = \text{Laplacian} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\left.\begin{matrix} x_x, & y_y, & z_z \\ x_y, & x_z, & y_z \end{matrix}\right\} \text{unit strain components}$$

$$\left.\begin{matrix} X_x, & Y_y, & Z_z \\ X_y, & X_z, & Y_z \end{matrix}\right\} \text{unit stresses}$$

7.8. Stress-Strain Equations for Isotropic Media

$$x_x = \frac{1}{E}(X_x - \sigma Y_y - \sigma Z_z)$$

$$y_y = \frac{1}{E}(Y_y - \sigma X_x - \sigma Z_z)$$

$$z_z = \frac{1}{E}(Z_z - \sigma X_x - \sigma Y_y)$$

$$\text{shear strain} = \frac{1}{A} \times \text{shear stress}$$

$$X_x = 2\left(\sigma B\epsilon + A\frac{\partial u}{\partial x}\right)$$

$$Y_y = 2\left(\sigma B\epsilon + A\frac{\partial v}{\partial y}\right)$$

$$Z_z = 2\left(\sigma B\epsilon + A\frac{\partial w}{\partial z}\right)$$

$$X_y = A\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$X_z = A\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$Y_z = A\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

For plane stress in xy plane

$$x_x = \frac{1}{E}(X_x - \sigma Y_y)$$

$$y_y = \frac{1}{E}(Y_y - \sigma X_x)$$

$$x_y = \frac{1}{A}X_y$$

$$X_x = \frac{E}{1-\sigma^2}(x_x + \sigma y_y)$$

$$Y_y = \frac{E}{1-\sigma^2}(y_y + \sigma x_x)$$

$$X_y = A x_y$$

7.9. Mathematical Derivations

7.91. *Longitudinal Vibrations in Two-Dimensional Plates*

As explained in the text, solutions for the infinite strip of Fig. 7.7 are first derived. Let

$$\left.\begin{aligned} u &= U \cos kx \cos \ell_y \\ v &= V \sin kx \sin \ell_y \end{aligned}\right\} \tag{7.12}$$

where U and V are constant. From these expressions $\epsilon = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$ can be obtained and substituted into the equilibrium equations 7.2. Two expressions as follows result after dividing through by the common term $\cos kx \cos \ell_y$.

$$\begin{aligned} A(k^2 + \ell^2) - \frac{Bk}{U}(-kU + \ell V) &= \rho\omega^2 \\ A(k^2 + \ell^2) + \frac{B\ell}{V}(-kU + \ell V) &= \rho\omega^2 \end{aligned} \tag{7.25}$$

Subtracting the second from the first of these equations, it is seen that

$$\left(\frac{Bk}{U} + \frac{B\ell}{V}\right)(-kU + \ell V) = 0 \tag{7.26}$$

Either or both of these factors equal to zero will satisfy 7.26, so that two values of $\dfrac{V}{U}$ are obtained. By substituting back into equations (7.25), conditions on ω^2 are found. The two solutions will be

$$\begin{aligned} \frac{V_1}{U_1} &= -\frac{\ell_1}{k} \text{ with } (A + B)(k^2 + \ell_1^2) = \rho\omega^2 \\ \frac{V_2}{U_2} &= \frac{k}{\ell_2} \text{ with } A(k^2 + \ell_2^2) = \rho\omega^2 \end{aligned} \tag{7.27}$$

By superimposing the two solutions the u and v displacements now become

$$\begin{aligned} u &= [U_1 \cos \ell_1 y + U_2 \cos \ell_2 y] \cos kx \\ v &= \left[-\frac{\ell_1}{k} U_1 \sin \ell_1 y + \frac{k}{\ell_2} U_2 \sin \ell_2 y\right] \sin kx \end{aligned} \tag{7.28}$$

Using the relationships of Section 7.8, one may now calculate all stresses.

The argument k has purposely been kept the same for both of the superimposed solutions in order that boundary conditions at $y = \pm\frac{b}{2}$ might be satisfied regardless of x.

For Y_y to equal zero at the edges of the strip

$$\frac{\partial v}{\partial y} + \sigma \left.\frac{\partial u}{\partial x}\right|_{y=\pm\frac{b}{2}} = 0 \tag{7.29}$$

This gives rise to the equation:

$$U_1(\ell_1^2 + \sigma k^2) \cos \ell_1 \frac{b}{2} - U_2[k^2(1 - \sigma)] \cos \ell_2 \frac{b}{2} = 0 \tag{7.30}$$

Similarly, if $X_y = 0$ at $y = \pm\frac{b}{2}$

$$\frac{\partial u}{\partial y} + \left.\frac{\partial v}{\partial x}\right|_{y=\pm\frac{b}{2}} = 0 \tag{7.31}$$

Another relation is obtained from (7.31):

$$-2\ell_1 \ell_2 U_1 \sin \ell_1 \frac{b}{2} + U_2(k^2 - \ell_2^2) \sin \ell_2 \frac{b}{2} = 0 \tag{7.32}$$

The two equations (7.30) and (7.32) will be satisfied if the determinant of the coefficients of U_1 and U_2 vanishes. The following transcendental equation will then be obtained; values of ℓ_1^2 and ℓ_2^2 being those required by Eq. 7.27.

$$\frac{\cot \ell_1 \frac{b}{2}}{\cot \ell_2 \frac{b}{2}} = \frac{-2\ell_1 \ell_2 k^2(1 - \sigma)}{(\ell_2^2 - k^2)(\ell_1^2 + \sigma k^2)} \tag{7.13}$$

By using either of equations (7.30) and (7.32), one may derive the relation between U_2 and U_1 provided a solution to (7.13) is found.

$$U_2 = U_1 \frac{-2\ell_1 \ell_2 \sin \ell_1 \frac{b}{2}}{(\ell_2^2 - k^2) \sin \ell_2 \frac{b}{2}} \tag{7.33}$$

To solve equation (7.13) assume a value for k^2 and plot graphically the right and left hand expressions as functions of $\theta^2 = \frac{\rho\omega^2}{A}$. Roots are indicated

by the crossover points. Values of θ^2 corresponding to different values of k^2 may also be found in this way and a curve plotted for θ^2 versus k^2, (or for $\theta \cdot b$ versus $k \cdot b$).

7.92. *Thickness Shear Vibrations*

To obtain a formula for the approximate resonant frequencies of thickness shear for a plate having large ratios of $\frac{W}{T}$ and $\frac{L}{T}$, one may consider the following displacements:

$$\begin{aligned} u &= U \sin kx \cos \ell_y \cos rz \\ v &= 0 \\ w &= 0 \end{aligned} \tag{7.34}$$

If there are no cross couplings between shear stresses or between shear and extensional stresses one may write:[8]

$$\begin{aligned} X_x &= c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} = c_{11}\, kU \cos kx \cos \ell y \cos rz \\ X_y &= c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -c_{66}\, \ell U \sin kx \sin \ell y \cos rz \\ X_z &= c_{55}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = -c_{55}\, rU \sin kx \cos \ell y \sin rz \end{aligned} \tag{7.35}$$

Substituting into the first of the equilibrium equations (7.1) and dividing through by common factors

$$c_{11}\, k^2 + c_{66}\, \ell^2 + c_{55}\, r^2 = \rho\omega^2 \tag{7.36}$$

The other two of equations (7.1) may be neglected if k and r are quite small so that it will only be necessary to consider equation (7.36) which can be solved for ω^2. It will be noticed that the X_y shear stress will predominate under these restrictions on k and r. Letting $k = \frac{n\pi}{L}$, $\ell = \frac{m\pi}{T}$, and $r = \frac{p\pi}{W}$ (n, m and p are integers) in order to satisfy the boundary conditions for this stress and also for X_z, one obtains the following formula. (This choice of k, ℓ, and r is also required if the shear stress is to vary in essentially the same manner as is experimentally observed.)

$$\omega = 2\pi f = \pi\sqrt{\frac{1}{\rho}}\sqrt{\frac{c_{11}n^2}{L^2} + \frac{c_{66}m^2}{T^2} + \frac{c_{55}p^2}{W^2}} \tag{7.17}$$

in which L and W must be much larger than T.

[8] Refer to equation (7.18) for values of c constants for isotropic case.

The boundary condition for the extensional stresses will not be met; however, they will be quite small in comparison to X_y if k is small, and may be neglected.

7.93. *Thickness Flexures*

Consider a three dimensional plate having a thickness b lying along the y direction. The following displacements are found to be of a form that can be made to satisfy the equilibrium equations 7.2.

$$\left.\begin{aligned} u &= U \sin kx \sin \ell y \cos mz \\ v &= V \cos kx \cos \ell y \cos mz \\ w &= W \cos kx \sin \ell y \sin mz \end{aligned}\right\} \tag{7.37}$$

Performing the operations indicated and substituting into the equilibrium equations give the following result:

$$\left.\begin{aligned} A(k^2 + \ell^2 + m^2) + \frac{Bk}{U}(kU - \ell V + mW) &= \rho\omega^2 \\ A(k^2 + \ell^2 + m^2) - \frac{B\ell}{V}(kU - \ell V + mW) &= \rho\omega^2 \\ A(k^2 + \ell^2 + m^2) + \frac{Bm}{W}(kU - \ell V + mW) &= \rho\omega^2 \end{aligned}\right\} \tag{7.38}$$

Subtract the second and third equations of (7.38) from the first:

$$\begin{aligned} &\text{then} \qquad \frac{Bk}{U} + \frac{B\ell}{V} = 0 \quad \text{and} \quad \frac{Bk}{U} - \frac{Bm}{W} = 0 \\ &\text{or} \qquad \frac{V}{U} = -\frac{\ell}{k} \quad \text{and} \quad \frac{W}{U} = \frac{m}{k} \end{aligned} \tag{7.39}$$

Putting these values back into 7.38, it is seen that the following relationship must be satisfied.

$$(A + B)(k^2 + \ell^2 + m^2) = \rho\omega^2 \tag{7.40}$$

Letting $\frac{V}{U} = -\frac{\ell}{k}$ as in (7.39), another value for $\frac{W}{U}$ may be obtained. The first and second equations of (7.38) will be satisfied for any ratio of $\frac{W}{U}$, so the 3rd equation is used.

$$A(k^2 + \ell^2 + m^2) + B\left(k^2 + \ell^2 + mk\frac{W}{U}\right) = \rho\omega^2 \tag{7.41}$$

Solving for $\frac{W}{U}$, using (7.41) and first of equations (7.38)

$$\frac{W}{U} = \frac{m}{k} \quad \text{and} \quad \frac{W}{U} = \frac{-k^2 - \ell^2}{mk} \tag{7.42}$$

The first ratio is the same as (7.39). For the second solution to the equilibrium equations, then the following relationships exist.

$$\frac{V}{U} = \frac{-\ell}{k} \quad \text{and} \quad \frac{W}{U} = \frac{-k - \ell^2}{mk} \tag{7.43}$$

When the above are substituted back into (7.38), it is found that[9]

$$A(k^2 + \ell^2 + m^2) = \rho\omega^2 \tag{7.44}$$

In a similar way, using $\frac{W}{U} = \frac{m}{k}$ the following are obtained:

$$\frac{W}{U} = \frac{m}{k}; \qquad \frac{V}{U} = \frac{k^2 + m^2}{k\ell} \tag{7.45}$$

with $A(k^2 + \ell^2 + m^2) = \rho\omega^2$ as before. This is the second solution for $\epsilon = 0$.

The three different solutions may now be combined or superimposed to give

$$u = [U_1 \sin \ell_1 y + U_2 \sin \ell_2 y + U_3 \sin \ell_3 y] \sin kx \cos mz$$

$$v = \left[-U_1 \frac{\ell_1}{k} \cos \ell_1 y - U_2 \frac{\ell_2}{k} \cos \ell_2 y + \frac{U_3(k^2 + m^2)}{\ell_3 k} \cos \ell_3 y \right] \cos kx \cos mz \tag{7.46}$$

$$w = \left[U_1 \frac{m}{k} \sin \ell_1 y - U_2 \frac{(k^2 + \ell_2^2)}{mk} \sin \ell_2 y + U_3 \frac{m}{k} \sin \ell_3 y \right] \cos kx \sin mz$$

In the above equations $\ell_2^2 = \ell_3^2$ because of the double requirement of 7.44.[10]

It is now possible to calculate the stresses existing at any point. It is desired to choose U_1, U_2, and U_3 in such a manner that the boundary conditions at the two major surfaces of the plate are satisfied. By using the relations given in Section 7.8, the extensional stress Y_y, and the two shear stresses X_y and Y_z are calculated with the use of 7.46. They are then

[9] It should also be noticed that $\epsilon = 0$ for this solution.

[10] $k_1 = k_2 = k_3 = k$
$m_1 = m_2 = m_3 = m$

set to zero at the faces of the plate; i.e. at $y = \pm\frac{b}{2}$. Three equations, after simplifying, result.

For $Y_y = 0$ at $y = \pm\frac{b}{2}$

$$U_1\left[\sigma B(k^2 + \ell_1^2 + m^2)\sin\ell_1\frac{b}{2} + A\ell_1^2\sin\ell_1\frac{b}{2}\right] + U_2\left[A\ell_2^2\sin\ell_2\frac{b}{2}\right] - U_3\left[A(k^2+m^2)\sin\ell_3\frac{b}{2}\right] = 0 \tag{7.47}$$

For $X_y = 0$ at $y = \pm\frac{b}{2}$

$$U_1\left[2\ell_1\cos\ell_1\frac{b}{2}\right] + U_2\left[2\ell_2\cos\ell_2\frac{b}{2}\right] + U_3\left[\ell_3 - \frac{(k^2+m^2)}{\ell_3}\right]\left[\cos\ell_3\frac{b}{2}\right] = 0 \tag{7.48}$$

For $Y_z = 0$ at $y = \pm\frac{b}{2}$

$$U_1\left[2\ell_1 m\cos\ell_1\frac{b}{2}\right] + U_2\left[\left(\ell_2 m - \frac{\ell_2}{m}(\ell_2^2 + k^2)\right)\cos\ell_2\frac{b}{2}\right] + U_3\left[\left(\ell_3 m - \frac{m}{\ell_3}(k^2+m^2)\right)\cos\ell_3\frac{b}{2}\right] = 0 \tag{7.49}$$

In order for these three equations to be satisfied simultaneously a necessary condition is that the third order determinant formed by the coefficients of the U's vanish. That is,

$$\begin{vmatrix} \left[(\sigma B(k^2+\ell_1^2+m^2) + A\ell_1^2)\sin\ell_1\frac{b}{2}\right] & \left[A\ell_2^2\sin\ell_2\frac{b}{2}\right] & -\left[A(k^2+m^2)\sin\ell_3\frac{b}{2}\right] \\ \left[2\ell_1\cos\ell_1\frac{b}{2}\right] & \left[2\ell_2\cos\ell_2\frac{b}{2}\right] & \left[\left(\ell_3 - \left(\frac{k^2+m^2}{\ell_3}\right)\right)\cos\ell_3\frac{b}{2}\right] \\ \left[2\ell_1 m\cos\ell_1\frac{b}{2}\right] & \left[\left(\ell_2 m - \frac{\ell_2}{m}(\ell_2^2+k^2)\right)\cos\ell_2\frac{b}{2}\right] & \left[\left(\ell_3 m - \frac{m}{\ell_3}(k^2+m^2)\right)\cos\ell_3\frac{b}{2}\right] \end{vmatrix} = 0 \tag{7.50}$$

By dividing row 1 by $\cos \ell_1 \frac{b}{2}$, row 2 by $\cos \ell_2 \frac{b}{2}$; row 3 by $\cos \ell_3 \frac{b}{2}$ and by subtracting the elements of row 3 from those of row 2, considerable simplification results.

The full expansion gives tbe following equation, after further simplifying operations are performed.

$$\left[(\sigma B(k^2 + \ell_1^2 + m^2) + A\ell_1^2) \tan \ell_1 \frac{b}{2}\right] \cdot [\ell_3^2 - k^2 - m^2] + 2\ell_1 \ell_3 \left[A(k^2 + m^2) \tan \ell_3 \frac{b}{2}\right] = 0 \tag{7.51}$$

It should be noticed that ℓ_2 bas dropped out entirely. Actually $\ell_2 = \ell_3$ as previously cxplained. Also the expression

$$\left[\ell_2 m + \frac{\ell_2}{m}(\ell_2^2 + k^2)\right]$$

must not be zero, for in simplifying equation (7.51) it was used as a divisor.

Equation (7.51) may be rewritten to give

$$\frac{\tan \ell_1 \frac{b}{2}}{\tan \ell_3 \frac{b}{2}} = \frac{-2\ell_1 \ell_3 A(k^2 + m^2)}{[\sigma B(k^2 + \ell_1^2 + m^2) + A\ell_1^2][\ell_3^2 - k^2 - m^2]} \tag{7.52}$$

In the above

$$\left.\begin{aligned} (A + B)(k^2 + \ell_1^2 + m^2) &= \rho\omega^2 \\ A(k^2 + \ell_3^2 + m^2) &= \rho\omega^2 \end{aligned}\right\} \tag{7.53}$$

By letting $\theta^2 = \frac{\rho\omega^2}{A}$ and $k^2 + m^2 = \alpha^2$ equations (7.52) and (7.53) above become

$$\frac{\tan \ell_1 \frac{b}{2}}{\tan \ell_3 \frac{b}{2}} = \frac{-2\ell_1 \ell_3 A\alpha^2}{[\sigma B(\ell_1^2 + \alpha^2) + A\ell_1^2] \cdot [\ell_3^2 - \alpha^2]} \tag{7.22}$$

witb

$$\ell_1^2 = \theta^2 \cdot \frac{A}{A + B} - \alpha^2$$

$$\ell_2^2 = \ell_3^2 = \theta^2 - \alpha^2$$

Equation (7.22) represents the general solution for normal thickness vibrations in an isotropic plate of finite thickness extending to infinity in both major directions. Tbe analogy for plates of finite dimensions is considered in the text.